AN EXTENDED MAGNETOHYDRODYNAMICS MODEL FOR RELATIVISTIC WEAKLY COLLISIONAL PLASMAS

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ABSTRACT

Black holes that accrete far below the Eddington limit are believed to accrete through a geometrically thick, optically thin, rotationally supported plasma that we will refer to as a radiatively inefficient accretion flow (RIAF). RIAFs are typically collisionless in the sense that the Coulomb mean free path is large compared to GM/c^2, and relativistically hot near the event horizon. In this paper we develop a phenomenological model for the plasma in RIAFs, motivated by the application to sources such as Sgr A* and M87. The model is derived using Israel-Stewart theory, which considers deviations up to second order from thermal equilibrium, but modified for a magnetized plasma. This leads to thermal conduction along magnetic field lines and a difference in pressure, parallel and perpendicular to the field lines (which is equivalent to anisotrotropic viscosity). In the non-relativistic limit, our model reduces to the widely used Braginskii theory of magnetized, weakly collisional plasmas. We compare our model to the existing literature on dissipative relativistic fluids, describe the linear theory of the plasma, and elucidate the physical meaning of the free parameters in the model. We also describe limits of the model when the conduction is saturated and when the viscosity implies a large pressure anisotropy. In future work, the formalism developed in this paper will be used in numerical models of RIAFs to assess the importance of non-ideal processes for the dynamics and radiative properties of slowly accreting black holes.

1. INTRODUCTION AND ASTROPHYSICAL CONTEXT

Most massive galaxies have black holes at their centers, and most of these black holes are accreting far below the Eddington rate $\dot{M}_{Edd}$ (Ho 2009). Low luminosity black holes are believed to accrete through a geometrically thick, optically thin disk. Phenomenological (radiatively inefficient accretion flows or RIAFs, see Yuan & Narayan 2014) and numerical (general relativistic magnetohydrodynamics, or GRMHD, see e.g. Koide et al. 1999; De Villiers et al. 2003; McKinney & Gammie 2004) models suggest that the density and temperature of the accreting plasma are such that the collisional (Coulomb) mean free paths (ion-ion, ion-electron, and electron-electron) are many orders of magnitude larger than GM/c^2 when $\dot{M} \ll \dot{M}_{Edd}$ (Mahadevan & Quataert 1997). The accreting plasma is thus *collisionless*.

In the nearby universe, the roster of low luminosity

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black holes includes M87 (accreting 4-5 orders of magnitude below $\dot{M}_{Edd}$), and Sgr A* (accreting about 8 orders of magnitude below $\dot{M}_{Edd}$). These two sources are the largest known black holes in terms of angular size on the sky. As a result, they are the two main targets for high resolution imaging experiments, including Gravity on the VLT (Eisenhauer et al. 2008) and the Event Horizon Telescope (EHT; Doeleman et al. 2009); the latter will use submillimeter very long baseline interferometry (VLBI) to resolve the accretion flow and jet on angular scales comparable to the event horizon. The EHT data may be used to test General Relativity by measuring the angular size of the photon orbit, if astrophysical uncertainties can be controlled (Psaltis et al. 2014).

The fact that the collisional mean free path is much larger than GM/c^2 in RIAFs implies that non-ideal processes such as conduction and viscosity are likely to be important. Furthermore, the mean free path is also much larger than the Larmor radii of all the species in the plasma and the gyration time scale is much shorter than the dynamical time scale. This leads to the above dissipative processes being anisotropic with respect to the local magnetic field. Heat flows only along the field lines

and an anisotropic viscosity is generated by a shear flow projected along the field lines.

Consider conduction, using non-relativistic estimates: the volume heating rate is $du/dt = -\boldsymbol{\nabla}\cdot\mathbf{q}$, where $\mathbf{q}$ is the conductive heat flux. The timescale for changing the internal energy is $\tau_{cond} = u/(du/dt)$. Estimate $\boldsymbol{\nabla}\cdot\mathbf{q} \sim q/r$, assume the heat flux is approximately saturated (given the large mean free paths) so that $q \sim uv$, where u is the internal energy density and v is rms particle speed. Then $\tau_{cond} \sim r/c$ for electrons (electrons are relativistic close to the horizon, so $v \sim c$) and $\tau_{cond} \sim r/c_s$ for protons, which in a near-virial RIAF is the dynamical time. Conduction can thus potentially play an important role in controlling the thermal state of the accreting plasma (e.g., Johnson & Quataert 2007).

What about viscosity or equivalently (as we show below) a difference in pressures along and perpendicular to the local magnetic field? Pressure anisotropy can be generated in a number of ways, for example, by anisotropic compression or expansion of the plasma, through a linear shear, etc. If the magnetic field strength varies, adiabatic invariance of the magnetic moment associated with the orbit of charged particles about a magnetic field with strength B implies that $T_\perp/B$ is invariant for a plasma with $kT \lesssim mc^2$ ($T_\perp \equiv$ temperature perpendicular to the local magnetic field). Here, the temperature T and the mass m corresponds to a specific species. Thus, if the plasma is compressed in the plane perpendicular to the mean field so that the density increases by a factor R, the perpendicular temperature also increases by a factor R, generating a significant pressure anisotropy, i.e., viscosity. Order unity fluctuations in density and magnetic field strength are common in numerical models of accretion flows, so order unity pressure anisotropy is expected in the absence of collective effects. This implies that viscous stresses may be dynamically important and contribute significantly to angular momentum transport and plasma heating (Sharma et al. 2006, 2007).

Despite their potential importance, conduction and viscosity are, however, absent from all global relativistic numerical models of RIAFs to date. This is one of the significant systematic uncertainties in developing models for the emission from systems such as Sgr A* and M87.

In this paper, we develop a formalism for modeling relativistic anisotropic conduction and viscosity, motivated by the application to RIAFs. Although the plasmas of interest are macroscopically collisionless, we focus in this paper on the more modest task of developing a theory for collisional magnetized plasmas in which the mean free path is large compared to the Larmor radius of particles, but small compared to the system size. The former hierarchy implies that heat and momentum transport are predominantly along the local magnetic field direction, while the latter constraint implies that one can derive the relevant equations using an expansion about thermal equilibrium. Our assumed hierarchy of length-scales is similar to that used in Braginskii 1965's theory of non-relativistic magnetized plasmas, which has been widely applied to understand the physics of dilute astrophysical plasmas (see Kulsrud 2005). We will refer to our formalism as an extended MHD (EMHD) model.

Although the applications that motivate this work are to collisionless systems, we focus on the collisional regime for the following reasons: (1) the theory of dissipative relativistic fluids is quite subtle (e.g. Andersson & Comer 2007), so it seems prudent to not jump directly to the yet more challenging long mean free path regime; (2) wave-particle interactions and velocity space instabilities limit the mean free path of charged particles to be much less than the collisional mean free path under the conditions of interest (Sharma et al. 2006; Kunz et al. 2014; Riquelme et al. 2015) implying that the 'collisional' theory may be more appropriate than one might have first anticipated. The formalism we develop allows the viscosity and conductivity to depend arbitrarily on local plasma conditions so that these wave-particle limits on the mean free path can be incorporated as sub-grid models.

Throughout the paper we formulate the equations in terms of a single fluid model. In reality, the low-collisionality plasmas of interest are believed to develop a two-temperature structure because the timescale for Coulomb collisions to equilibrate the electron and proton temperatures is long compared to the dynamical time in the accreting plasma. A formulation for dealing with electron dynamics and its numerical implementation has been recently introduced by Ressler et al. (2015), where a reduced form of our conduction model has been used and appropriately modified for electrons.

The remainder of this paper is organized as follows. In §2 we write down basic equations and describe the equivalence of viscosity and anisotropic pressure. In §3 we describe the desired asymptotic behavior of any extended MHD closure model. In §4 we derive the evolution equations for the heat flux and pressure anisotropy. §5 describes the connection between our model and non-relativistic dissipative theory. §6 motivates a scheme for fixing the model parameters (the transport coefficients : viscosity and conductivity) in terms of a relaxation time. §7 gives the linear theory and the stability thresholds of our model, and §8 a brief discussion of nonlinear (shock) solutions. Finally in §9 we offer a guide to the model, a summary of the formalism, and the relationship to earlier works. In the Appendix we show how the characteristic pressure anisotropy derived on thermodynamic grounds using the Israel-Stewart theory can also be interpreted as arising from conservation of relativistic adiabatic invariants in a magnetized plasma.

2. PHYSICAL CONTEXT

We begin by defining notation and frames.

We work in a spacetime described by the metric $g_{\mu\nu}$, whose determinant we denote with g. Consider a plasma consisting of particles with distribution function $f_s \equiv dN/d^3x\,d^3p$, and rest mass m_s, where $d^3p = dp_1 dp_2 dp_3$, p_i are the spatial *covariant* components of the particle four-momentum p_μ and s indicates the species of the particles (electrons, ions, etc.). The distribution function is invariant. Each species has a number current

$$N_s^\mu \equiv \int \frac{d^3p}{\sqrt{-g}\,p^t} \, p^\mu f_s. \qquad (1)$$

We assume that the plasma consists of electrons and ions, and is quasi-neutral everywhere. Thus both these species have the same approximate number current $N^\mu \equiv N_i^\mu \approx N_e^\mu$. We define the rest frame as that in which the num-

ber current has no spatial components. Therefore we have,

$$u^\mu \equiv \frac{N^\mu}{n}. \qquad (2)$$

where n is the number density of ions, and is equal to the number density of electrons. This definition of u^μ implies that we are using the so-called Eckart frame, in which mass diffusion is absent. An alternative, the Landau frame, assumes that energy diffusion is absent. The total rest mass density ρ, often denoted as ρ_0 in the relativity literature is $\rho = -\Sigma_s m_s N_s^\mu u_\mu = -(m_i N_i^\mu u_\mu + m_e N_e^\mu u_\mu)$. Since $N_i^\mu \approx N_e^\mu \equiv N^\mu$, we have $\rho = -(m_i + m_e)N^\mu u_\mu = (m_i + m_e)n$.

The matter stress-energy tensor is

$$T_{\text{matter}}^{\mu\nu} \equiv \sum_s \int \frac{d^3p}{\sqrt{-g}p^t} \, p^\mu p^\nu f_s. \qquad (3)$$

Each of these definitions is invariant because $d^3p/(\sqrt{-g}p^t)$ is invariant.

On taking moments of the Boltzmann equation, one can show that the quantities N^μ and $T_{\text{matter}}^{\mu\nu}$ satisfy

$$\nabla_\mu N^\mu = \nabla_\mu(nu^\mu) = 0 \qquad (4)$$

and

$$\nabla_\mu T_{\text{matter}}^{\mu\nu} = F^{\mu\nu} J_\mu \qquad (5)$$

where ∇_μ is the covariant derivative, $F^{\mu\nu}$ is the electromagnetic field tensor and J_μ is the four-current. The divergence of the electromagnetic stress tensor is

$$\nabla_\mu T_{\text{EM}}^{\mu\nu} = -F^{\mu\nu} J_\mu \qquad (6)$$

On adding (5) and (6), we get that the divergence of the total stress tensor $T^{\mu\nu}$ is zero, as required by the Bianchi identities

$$\nabla_\mu (T_{\text{matter}}^{\mu\nu} + T_{\text{EM}}^{\mu\nu}) = \nabla_\mu T^{\mu\nu} = 0 \qquad (7)$$

For a perfect unmagnetized fluid the stress-energy tensor is

$$T_{\text{matter}}^{\mu\nu} = (\rho + u)u^\mu u^\nu + Ph^{\mu\nu}, \qquad (8)$$

where

$$h^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu \qquad (9)$$

is the projection tensor (projects into the space normal to the fluid four-velocity), $g^{\mu\nu}$ is the metric, u the internal energy per unit proper volume, and P the gas pressure.

We define the *fluid frame* as an orthonormal tetrad with time component $e_{(t)}^\mu = u^\mu$, and three additional spacelike basis vectors $e_{(x)}^\mu, e_{(y)}^\mu$, and $e_{(z)}^\mu$. In the fluid frame the stress-energy tensor is

$$T_{\text{matter}}^{(a)(b)} = \begin{pmatrix} \rho + u & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}, \qquad (10)$$

which, with (3), provides a kinetic theory definition for the pressure and internal energy.

The space-space part of the stress-energy tensor is the pressure, or stress, tensor

$$P^{\mu\nu} \equiv h_\alpha^\mu h_\beta^\nu T^{\alpha\beta}. \qquad (11)$$

In the fluid frame, the spatial components of the ideal fluid pressure tensor are

$$P^{ij} = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}. \qquad (12)$$

In what follows we are interested in modeling a magnetized plasma that departs from ideality in that it has a conductive heat flux and a viscous stress. The total stress tensor $T^{\mu\nu}$ with the fluid assumed perfect and the electromagnetic terms included under the ideal magnetohydrodynamics (MHD) approximation (conductivity $\sigma = \infty$) is

$$T^{\mu\nu} = (\rho + u + \frac{1}{2}b^2)u^\mu u^\nu + (P + \frac{b^2}{2})h^{\mu\nu} - b^\mu b^\nu. \quad (13)$$

where $b^2 = b^\mu b_\mu$ and

$$b^\mu = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda} u_\nu F_{\lambda\kappa} \qquad (14)$$

where $\epsilon \equiv$ Levi-Civita tensor, which is antisymmetric on all pairs of indices. Evidently $b^\mu u_\mu = 0$, and b^μ reduces to the magnetic field in the fluid frame (with a factor of $\sqrt{4\pi}$ absorbed into the definition). The magnetic field evolution is given by

$$\nabla_\mu (u^\mu b^\nu - b^\mu u^\nu) = 0 \qquad (15)$$

which combines the induction equation (three space components) with the no-monopoles condition (time component).

When including non-ideal effects in the stress-energy tensor, the heat flux q^μ in the system is

$$q^\mu \equiv -h_\alpha^\mu u_\beta T^{\alpha\beta}. \qquad (16)$$

Combined with (3), this provides a kinetic theory definition for the heat flux. Evidently $u^\mu q_\mu = 0$, so in the fluid frame, $q^{(t)} = 0$. From (16) one can show that the heat flux makes a contribution to the stress tensor of the form

$$T_{\text{cond}}^{\mu\nu} = u^\mu q^\nu + u^\nu q^\mu, \qquad (17)$$

which in the fluid frame has the form

$$T_{\text{cond}}^{(a)(b)} = \begin{pmatrix} 0 & q^x & q^y & q^z \\ q^x & 0 & 0 & 0 \\ q^y & 0 & 0 & 0 \\ q^z & 0 & 0 & 0 \end{pmatrix}. \qquad (18)$$

The viscous stress tensor $\Pi^{\mu\nu}$ models momentum fluxes set up by departures from equilibrium due to a shear flow. It is given by

$$\Pi^{\mu\nu} + Ph^{\mu\nu} \equiv h_\alpha^\mu h_\beta^\nu T^{\alpha\beta}, \qquad (19)$$

It is perhaps not as widely appreciated as it should be, that the viscous stress can be recast as a pressure anisotropy. In the fluid frame the viscous stress tensor τ^{ij} is a symmetric matrix, so there is always a basis (obtained by rotation) where it can be written in diagonal form:

$$P^{ij} = \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_y & 0 \\ 0 & 0 & P_z \end{pmatrix}, \qquad (20)$$

where there is a separate pressure for each direction. Below we consider a magnetized plasma where the stress tensor is symmetric under rotations around the mag-

netic field. If the field is in the z direction, this implies $P_x = P_y$.

With conduction and viscosity of the plasma included, the total stress-energy tensor is now

$$T^{\mu\nu}_{\text{matter+EM}} = T^{\mu\nu} = (\rho + u + \tfrac{1}{2}b^2)u^\mu u^\nu + (P + \tfrac{1}{2}b^2)h^{\mu\nu} - b^\mu b^\nu + q^\mu u^\nu + q^\nu u^\mu + \Pi^{\mu\nu}. \tag{21}$$

This stress-energy tensor is quite general, but one needs an appropriate model for u, P, q^μ, and $\tau^{\mu\nu}$. Note that the electromagnetic terms are still written down in the infinite conductivity limit.

3. MODEL DESIDERATA

What are the desirable properties of a closure model for the heat flux q^μ and the viscous stress $\Pi^{\mu\nu}$?

(1) The model should be causal. For non-relativistic shear viscosity and thermal conduction, the energy and momentum fluxes are proportional to gradients of the temperature and velocity, and so respond instantaneously to changes in the fluid. Classical, non-relativistic models are parabolic and have characteristics that propagate at infinite speed. The classical theories can be made causal in a model pioneered by Maxwell and Cattaneo (1948) in which the energy and momentum fluxes relax to their classical values on a characteristic timescale τ.

(2) The model should be stable. The relativistic thermal conduction model of Eckart (1940) (see also MTW) sets

$$q^\mu = -\rho\chi\, h^{\mu\nu}\left(\partial_\nu\Theta + \Theta a_\nu\right) \tag{22}$$

where χ is the thermal diffusivity, $\Theta = kT/mc^2$ is the normalized temperature and a_ν is the four-acceleration. The term proportional to the four-acceleration drives the temperature toward a constant redshifted temperature rather than a constant local temperature - a desirable effect - but it makes the theory unstable (Garcia-Perciante et al. 2009; Lopez-Monsalvo & Andersson 2011). Long wavelength modes ($k \to 0$) are unstable with growth rate

$$\omega = \frac{(\rho c^2 + u + P)c^2}{\chi P} \tag{23}$$

Notice that as $\chi \to 0$, $\omega \to \infty$. If this theory were correct, the water in our bodies would spontaneously explode in 10^{-34} sec (Hiscock & Lindblom 1985). Evidently the stability of relativistic conduction theories is nontrivial. A relativistic extension of the Maxwell-Cattaneo procedure not only makes the theory hyperbolic but also conditionally eliminates the Eckart instability.

(3) Entropy should increase, i.e. the model should obey the second law of thermodynamics. The entropy constraint is expressed by defining an entropy four-current s^μ and requiring that $s^\mu_{;\mu} \geq 0$. This constraint was used to derive the Eckart model wherein the entropy current is expanded around equilibrium to first order in the heat flux. The first order model suffers from the instability described above. Expanding up to second order, as done by Israel & Stewart (1979) leads to conditionally hyperbolic, stable and causal equations. We will use this in the

next section to derive evolution equations for our model of anisotropic thermal conduction and viscosity.

(4) We are interested in plasmas with ion and electron Larmor radii tiny compared to the characteristic scale GM/c^2, and ion and electron gyroperiod tiny compared to the dynamical timescale. Therefore, we shall assume that the distribution functions of both the ions and electrons are independent of the gyrophase, i.e. $f_s \equiv f_s(p_\parallel, p_\perp)$, where s indicates the species. Now in a tetrad frame with $e^\mu_{(t)} = u^\mu$ and $e^\mu_{(z)} \equiv e^\mu_\parallel = \hat{b}^\mu$, we evaluate (3) to find only the following non-zero components

$$T^{(\mu)(\nu)}_{\text{matter}} = \begin{pmatrix} T^{(0)(0)} & 0 & 0 & T^{0\parallel} \\ 0 & T^{\perp\perp} & 0 & 0 \\ 0 & 0 & T^{\perp\perp} & 0 \\ T^{0\parallel} & 0 & 0 & T^{\parallel\parallel} \end{pmatrix}. \tag{24}$$

The terms which are zero are identically so, because they appear as $\int_0^{2\pi}\sin(\theta)d\theta$ or $\int_0^{2\pi}\cos(\theta)d\theta$, where θ is the gyrophase. We see that there is a heat flux $T^{0\parallel} \equiv q$ only along the magnetic field line. Therefore our model for the heat flux can be written as

$$q^\mu = q\,\hat{b}^\mu \tag{25}$$

where $\hat{b}^\mu = b^\mu/\sqrt{b^\mu b_\mu}$. q will be the fundamental variable describing the heat flux.[1]

We now write down the pressure tensor with $T_{\perp\perp} \equiv P_\perp$ and $T_{\parallel\parallel} \equiv P_\parallel$

$$P^{(i)(j)} = \begin{pmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{pmatrix} = \begin{pmatrix} P + \Delta P_\perp & 0 & 0 \\ 0 & P + \Delta P_\perp & 0 \\ 0 & 0 & P + \Delta P_\parallel \end{pmatrix}. \tag{26}$$

where P is the ideal fluid pressure and $\Delta P_\perp$ and $\Delta P_\parallel$ are deviations from it in the $\perp$ and $\parallel$ directions respectively. The variables $\Delta P_\perp$ and $\Delta P_\parallel$ can in principle vary independently and give rise to both a bulk viscosity (trace) and a shear viscosity (trace-free part). We simplify the model further by assuming that bulk viscosity is zero and thus imposing that the deviation of the pressure tensor from ideality be trace-free. Doing so gives $\Delta P_\parallel = -2\Delta P_\perp$. Now redefining $\Delta P_\perp \equiv \Delta P/3$, we have

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[1] A more accurate description of a collisionless plasma requires us to differentiate between a heat flow due to parallel temperatures gradients $q^\mu_\parallel$ and a heat flow due to perpendicular temperatures gradients $q^\mu_\perp$, both of which flow along the field lines $q^\mu_\parallel \equiv q_\parallel\,\hat{b}^\mu$ and $q^\mu_\perp \equiv q_\perp\,\hat{b}^\mu$. The net heat flow is then $q^\mu = q\,\hat{b}^\mu \equiv (q_\perp + q_\parallel)\,\hat{b}^\mu$. However, even in this case, the heat flux appears in the stress tensor only as the sum $q \equiv q_\perp + q_\parallel$.

for the pressure tensor

$$P^{(i)(j)} = \begin{pmatrix} P + \frac{1}{3}\Delta P & 0 & 0 \\ 0 & P + \frac{1}{3}\Delta P & 0 \\ 0 & 0 & P - \frac{2}{3}\Delta P \end{pmatrix}. \quad (27)$$

We see that $P_\perp = P + \Delta P/3$ and $P_\parallel = P - 2\Delta P/3$. Therefore $\Delta P = P_\perp - P_\parallel$, which is the usual definition of pressure anisotropy. The shear stress in an arbitrary frame is then

$$\Pi^{\mu\nu} = -\Delta P \left(\hat{b}^\mu \hat{b}^\nu - \frac{1}{3}h^{\mu\nu} \right). \quad (28)$$

ΔP is the fundamental variable describing the viscous stress. The above expression satisfies $\Pi^\mu_\mu = 0$ and is thus trace-free.

(5) If possible, the model should asymptote to a rigorous model in the collisional limit. The relation to earlier theories will be discussed in detail later, but in brief our model is equivalent to Israel & Stewart (1979) theory projected along the magnetic field lines. Israel-Stewart theory has 9 fields that describe nonideal effects: 5 for the shear viscosity, 1 for bulk viscosity, and 3 for the conductivity. Projecting along the magnetic field lines reduces the viscous shear stress degrees of freedom from 5 to 1, and the heat flux degrees of freedom from 3 to 1, while we ignore bulk viscosity.

4. EVOLUTION OF Q AND ΔP

Following Israel and Stewart, it is possible to derive evolution equations for q and ΔP from the second law of thermodynamics, expressed here by the requirement that the entropy current s^μ have positive divergence: $\nabla_\mu s^\mu \geq 0$.

First, what is the entropy current? In ideal hydrodynamics

$$s^\mu = s\rho u^\mu \quad (29)$$

where the entropy per baryon s depends on the equation of state $P = P(\rho, u)$. Here we assume

$$P = (\gamma - 1)u \quad (30)$$

and

$$P = \rho\Theta. \quad (31)$$

The first law then implies $ds = (du/u - \gamma d\rho/\rho)/(\gamma - 1)$. One can show that, if $\tau \equiv$ proper time,

$$0 = u_\nu \nabla_\mu T^{\mu\nu}_{\text{ideal}} = -P\frac{ds}{d\tau} \quad (32)$$

where $T^{\mu\nu}_{\text{ideal}}$ is the ideal gas stress-energy tensor. Combining this result with the continuity equation gives

$$\nabla_\mu s^\mu = 0. \quad (33)$$

In nonideal hydrodynamics one thinks of q^μ and $\Pi^{\mu\nu}$ as small corrections to the ideal model. Expanding to second order in these small corrections, the most general possible entropy current subject to the constraints $q_\mu u^\mu = 0$, $\Pi^\mu_\nu u^\nu = 0$ and $\Pi^\mu_\mu = 0$ is:

$$s^\mu = s\rho u^\mu + \frac{a_1}{\Theta}q^\mu - \frac{b_1}{2\Theta}q^\alpha q_\alpha u^\mu - \frac{b_2}{2\Theta}\Pi^{\alpha\beta}\Pi_{\alpha\beta}u^\mu - \frac{c_1}{2\Theta}q^\alpha \Pi^\mu_\alpha. \quad (34)$$

This is precisely what is done in Israel-Stewart theory, except that our bulk viscosity is 0 (and working in the

Eckart frame eliminates another term related to mass diffusion). The factors of $1/\Theta$ are chosen for convenience. The ordering in the above expansion is $|q^\mu| \sim |\Pi^{\mu\nu}| \sim \epsilon \ll 1$. The first term $s\rho u^\mu$ is the leading order term $O(\epsilon^0)$ and is present even in the ideal case. The term $\propto q^\mu$ is first order in a dissipative field $O(\epsilon)$, i.e. here the heat flux q^μ. The terms $\propto q^\alpha q_\alpha$, $\Pi^{\alpha\beta}\Pi_{\alpha\beta}$ and $q^\alpha \Pi^\mu_\alpha$ are second order $O(\epsilon^2)$. Note that there is no viscosity contribution at first order in the above expansion, as is explained in section (5.2).

We now set $c_1 = 0$ to simplify the model further. The neglected term couples q and ΔP. The value of c_1 cannot be determined at this thermodynamic level and one has to resort to kinetic theory (see section C in Bouras et al. 2010). However, its choice does not affect the amount of entropy production. We remark further on the effect of $c_1 \neq 0$ at the end of the derivation. Thus we have

$$s^\mu = s\rho u^\mu + \frac{a_1}{\Theta}q\hat{b}^\mu - \frac{b_1}{2\Theta}q^2 u^\mu - \frac{b_2}{3\Theta}\Delta P^2 u^\mu. \quad (35)$$

where we have used $\Pi^{\mu\nu}\Pi_{\mu\nu} = \frac{2}{3}\Delta P^2$. Now evaluate $\nabla_\mu s^\mu$. First,

$$\nabla_\mu(s\rho u^\mu) = \frac{1}{\Theta}u_\nu \nabla_\mu T^{\mu\nu}_{C+V} \quad (36)$$

where T_{C+V} is the sum of the conduction and viscosity terms in the stress-energy tensor. Then the conduction terms give

$$u_\nu \nabla_\mu(u^\mu q^\nu + q^\mu u^\nu) = -q^\mu a_\mu - \nabla_\mu q^\mu. \quad (37)$$

where $a_\mu \equiv u^\alpha \nabla_\alpha u_\mu$ is the four-acceleration, and we have used the constraint $u_\mu q^\mu = 0$. The viscosity terms give

$$u_\mu \nabla_\nu(-\Delta P(\hat{b}^\mu \hat{b}^\nu - \frac{1}{3}h^{\mu\nu})) = \Delta P \left(\hat{b}^\mu \hat{b}^\nu \nabla_\mu u_\nu - \frac{1}{3}\nabla_\mu u^\mu \right). \quad (38)$$

In deriving the above, we have used the constraint $u_\mu \hat{b}^\mu = 0 \Rightarrow u_\mu \hat{b}^\nu \nabla_\nu \hat{b}^\mu = -\hat{b}^\mu \hat{b}^\nu \nabla_\nu u_\mu$. Next, the first order term in q is

$$\nabla_\mu \left(\frac{a_1}{\Theta}q^\mu \right) = \frac{a_1}{\Theta}\nabla_\mu q^\mu + q^\mu \nabla_\mu \left(\frac{a_1}{\Theta} \right) \quad (39)$$

the second order term in q is

$$-\nabla_\mu \left(\frac{b_1}{2\Theta}q^2 u^\mu \right) = -\frac{b_1 q}{\Theta}\frac{dq}{d\tau} - \frac{q^2}{2}\nabla_\mu \left(\frac{b_1 u^\mu}{\Theta} \right) \quad (40)$$

where $dq/d\tau = u^\mu \nabla_\mu q$, and the second order term in ΔP is

$$-\nabla_\mu \left(\frac{b_2}{3\Theta}\Delta P^2 u^\mu \right) = -\frac{2b_2 \Delta P}{3\Theta}\frac{d\Delta P}{d\tau} - \frac{\Delta P^2}{3}\nabla_\mu \left(\frac{b_2 u^\mu}{\Theta} \right) \quad (41)$$

Assembling all the first-order terms in q from $\nabla_\mu s^\mu$,

$$\frac{a_1 - 1}{\Theta}\nabla_\mu q^\mu - q^\mu a_\mu + q^\mu \nabla_\mu \left(\frac{a_1}{\Theta} \right). \quad (42)$$

The term proportional to $\nabla_\mu q^\mu$ has indeterminate sign, so we choose $a_1 = 1$. Gathering all the terms (first order + second order) in q (and writing down only the q terms of $\nabla_\mu (s\rho u^\mu)$), we find

$$\nabla_\mu \left(s\rho u^\mu + \frac{q}{\Theta}\hat{b}^\mu - \frac{b_1}{2\Theta}q^2 u^\mu \right) = q^\mu \left[-\frac{\nabla_\mu\Theta}{\Theta^2} - \frac{a_\mu}{\Theta} - \frac{q_\mu}{2}\nabla_\alpha\left(\frac{b_1 u^\alpha}{\Theta}\right) - \frac{b_1\hat{b}_\mu}{\Theta}\frac{dq}{d\tau}\right].$$ (43)

Evidently this can be positive definite if the quantity in square brackets $= \beta_1 q_\mu$ and $\beta_1 > 0$. Remarkably, by applying this condition we find an evolution equation for q:

$$\frac{dq}{d\tau} = -\frac{\Theta}{b_1}\left(\beta_1 q + \frac{\hat{b}^\mu(\nabla_\mu\Theta + \Theta a_\mu)}{\Theta^2} + q\nabla_\alpha\left(\frac{b_1 u^\alpha}{2\Theta}\right)\right)$$ (44)

All that remains is to fix the constants b_1 and β_1. For β_1, we require that q asymptote to its non-relativistic value, $-\rho\chi\boldsymbol{\nabla}\Theta$, where χ is the conductive diffusivity and has dimensions of a length times a velocity. Then $\beta_1 = (\rho\chi\Theta^2)^{-1}$. For b_1, we require that it be proportional to a relaxation timescale τ_R. Then $b_1 = \tau_R/(\rho\chi\Theta)$. Gathering all together, our final evolution equation for the heat flux is

$$\frac{dq}{d\tau} = -\frac{q - q_0}{\tau_R} - \frac{q}{2}\frac{d}{d\tau}\log\left(\frac{\tau_R}{\chi P^2}\right)$$ (45)

where

$$q_0 \equiv -\rho\chi\hat{b}^\mu(\nabla_\mu\Theta + \Theta a_\mu)$$ (46)

is the classical Eckart heat flux projected onto the magnetic field. The heat flux q relaxes to the first order (Eckart) heat flux q_0 over a timescale τ_R. The additional term on the right is a second order correction and is formally necessary to ensure the positivity of entropy production. The importance of this term in a full calculation can only be gauged by performing a calculation with and without this term. Notice that equation (45) can be rewritten in the remarkably simple, scaled form

$$\frac{dQ}{d\tau} = -\frac{Q - Q_0}{\tau_R},$$ (47)

with $Q \equiv q(\tau_R/(\chi P^2))^{1/2}$.

Next, assemble all terms in $\nabla_\mu s^\mu$ depending on ΔP (and writing down only the ΔP terms of $\nabla_\mu\left(s\rho u^\mu\right)$) to find

$$\nabla_\mu\left(s\rho u^\mu - \frac{b_2}{3\Theta}\Delta P^2 u^\mu\right) = \Delta P\left[\frac{1}{\Theta}\left(\hat{b}^\mu\hat{b}^\nu\nabla_\mu u_\nu - \frac{1}{3}\nabla_\mu u^\mu\right) - \frac{\Delta P}{3}\nabla_\mu\left(\frac{b_2 u^\mu}{\Theta}\right) - \frac{2b_2}{3\Theta}\frac{d\Delta P}{d\tau}\right].$$ (48)

Evidently this will be positive definite if the quantity in square brackets $= \beta_2\Delta P$ and $\beta_2 > 0$. This provides an evolution equation for ΔP:

$$\frac{d\Delta P}{d\tau} = \frac{3\Theta}{2b_2}\left(-\beta_2\Delta P + \frac{1}{\Theta}(\hat{b}^\mu\hat{b}^\nu\nabla_\mu u_\nu - \frac{1}{3}\nabla_\mu u^\mu)\right) - \frac{\Theta\Delta P}{2b_2}\nabla_\mu\left(\frac{b_2 u^\mu}{\Theta}\right).$$ (49)

Again set the coefficients by requiring that b_2 be proportional to a relaxation time τ_R and that ΔP asymptote to its classical non-relativistic limit $= 3\rho\nu(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \boldsymbol{\nabla}\mathbf{v} - \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{v})$ (see the Appendix), with $\nu \equiv$ kinematic viscosity, to find $b_2 = \tau_R/(2\rho\nu)$ and $\beta_2 = (3\rho\nu\Theta)^{-1}$. Gathering all together, the final evolution equation for ΔP is

$$\frac{d\Delta P}{d\tau} = -\frac{\Delta P - \Delta P_0}{\tau_R} - \frac{\Delta P}{2}\frac{d}{d\tau}\log\left(\frac{\tau_R}{\rho\nu P}\right),$$ (50)

where

$$\Delta P_0 \equiv 3\rho\nu(\hat{b}^\mu\hat{b}^\nu\nabla_\mu u_\nu - \frac{1}{3}\nabla_\mu u^\mu)$$ (51)

is a covariant generalization of the Braginskii (1965)

model, in which collisions balance the forcing of anisotropy by the velocity field (see appendix A). As we already saw in (45) where the heat flux relaxes to Eckart theory over a timescale τ_R, the pressure anisotropy ΔP also, relaxes to its first order value ΔP_0, with the second term on the right hand side of (50) being a higher order correction required for positivity of the entropy production. Notice that equation (50) can also be written in simplified, scaled form:

$$\frac{dD}{d\tau} = -\frac{D - D_0}{\tau_R}$$ (52)

where $D \equiv \Delta P(\tau_R/(\rho\nu P))^{1/2}$.

It is also useful to gather the full, relativistic dissipation function:

$$\nabla_\mu s^\mu = \nabla_\mu\left(\rho s u^\mu + \frac{q}{\Theta}\hat{b}^\mu - \frac{\tau_R}{2\rho\chi\Theta^2}q^2 u^\mu - \frac{\tau_R}{6\rho\nu\Theta}\Delta P^2 u^\mu\right) = \frac{q^2}{\chi\rho\Theta^2} + \frac{1}{3}\frac{\Delta P^2}{\nu\rho\Theta}$$ (53)

which is positive definite, has the correct units, and   reduces to the correct dissipation function in the non-

relativistic limit as we shall show in the next section. Note that the right hand side of (53) is a function of the full heat flux q and the pressure anisotropy ΔP, that are solved for using (45) and (50) respectively, and not the relaxed forms q_0 and ΔP_0. The above value of entropy production is invariant under the choice of the cross coupling coefficient c_1 in (35). However, had we not set $c_1 = 0$, the evolution equations for q (45) and ΔP (50) would both have additional terms which couple q and ΔP to each other. [2]

The model given by equations (45) and (50) is derived using precisely the same procedure as the Israel-Stewart model, but the complexity is greatly reduced because – thanks to the magnetic field – there are only two non-ideal degrees of freedom. We will also see below that the model is not subject to the linear instabilities of the isotropic first order theory (where $b_1 = b_2 = 0$) discovered by Hiscock & Lindblom (1985), provided the damping timescale τ_R is chosen appropriately, as found by Hiscock & Lindblom (1983) and as we shall see in Sec. 7.

5. CONNECTION TO NON-RELATIVISTIC DISSIPATIVE THEORY

In this section we compare the equations governing the entropy scalar s and the entropy current s^μ (53) to their non-relativistic counterparts. It is important to stress that the entropy scalar s and the entropy current s^μ are distinct quantities which obey separate evolution equations. Both of these have an analog in non-relativistic dissipative hydrodynamics. In equilibrium, the two quantities are related by $s = -s^\mu u_\mu / \rho$, but this is not true in general. This can be seen explicitly in (35) where there are second order differences ($\sim q^2, \Delta P^2$) between s and $-s^\mu u_\mu / \rho$.

5.1. Entropy scalar s

In non-relativistic dissipative hydrodynamics, the evolution equation for the entropy scalar s (Landau & Lifshitz 1987) is

$$\rho\Theta \frac{Ds}{Dt} = -\boldsymbol{\nabla} \cdot \mathbf{q} - \boldsymbol{\Pi} : \boldsymbol{\nabla}\mathbf{u} \qquad (54)$$

where D/Dt is the convective derivative $\partial/\partial t + \mathbf{u} \cdot \boldsymbol{\nabla}$. We now derive the corresponding equation in relativistic dissipative hydrodynamics starting from (32), which itself has been derived from the first law of thermodynamics. Proceeding to do so

$$\rho\Theta \frac{ds}{d\tau} = -u_\nu \nabla_\mu T^{\mu\nu}_{\text{ideal}} = u_\nu \nabla_\mu T^{\mu\nu}_{C+V} \qquad (55)$$

$$= u_\nu \nabla_\mu \left(q^\mu u^\nu + q^\nu u^\mu + \Pi^{\mu\nu} \right) \qquad (56)$$

$$= -\nabla_\mu q^\mu - q^\mu a_\mu - \Pi^{\mu\nu}\nabla_\mu u_\nu \qquad (57)$$

where we have used (37) and the constraint $\Pi^{\mu\nu} u_\nu =$

$0 \Rightarrow u_\nu \nabla_\mu \Pi^{\mu\nu} = -\Pi^{\mu\nu}\nabla_\mu u_\nu$. The above equation is independent of the model for q^μ and $\Pi^{\mu\nu}$, and only uses the form of the dissipative component $T^{\mu\nu}_{C+V}$ of the stress-tensor. Thus, the equation is valid for both the first order Eckart theory as well as the second order Israel-Stewart theory. The difference between the non-relativistic equation (54) and the relativistic equation (57), apart from the 3-derivatives transforming into covariant derivatives, is the presence of the $q^\mu a_\mu$ term, which has no equivalent in the non-relativistic case.

5.2. Entropy current s^μ

The equation for the divergence of the entropy current (53) has been derived using a second order ansatz for the entropy current (35). However, the value of the entropy production is the same as in first order theories, two of which are the relativistic theory by Eckart, and the classical non-relativistic theory of dissipation. Below we show, starting from the evolution equation for the entropy scalar s from the previous subsection, that the entropy production in (53) is true for both the relativistic and non-relativistic first order theories.

- For the relativistic first order Eckart theory, we start with (57) and use the first order isotropic heat flux $q^\mu = q_0^\mu = -\rho\chi h^{\mu\nu}\left(\nabla_\mu\Theta + \Theta a_\mu\right)$, the first order shear tensor $\Pi_{\mu\nu} = \Pi_{0\mu\nu} = -2\rho\nu\langle\nabla^\mu u^\nu\rangle = -\rho\nu h^\alpha_\mu h^\beta_\nu\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - (2/3)h_{\alpha\beta}h^{\kappa\eta}\nabla_\kappa u_\eta\right)$ (Andersson & Comer 2007), along with the relativistic continuity equation (4) to get

$$\nabla_\mu\left(\rho s u^\mu + \frac{q_0^\mu}{\Theta}\right) = \frac{q_0^\mu q_{0\mu}}{\rho\chi\Theta^2} + \frac{\Pi_0^{\mu\nu}\Pi_{0\mu\nu}}{2\rho\nu\Theta} \qquad (58)$$

In the anisotropic case, we again start with (57), but now use the heat flux $q^\mu = q_0 \, \hat{b}^\mu$ and the shear stress $\Pi^{\mu\nu} = -\Delta P_0\left(\hat{b}^\mu\hat{b}^\nu - \frac{1}{3}h^{\mu\nu}\right)$, where q_0 and ΔP_0 are given by (46) and (51) respectively, along with (4) to get

$$\nabla_\mu\left(\rho s u^\mu + \frac{q_0}{\Theta}\hat{b}^\mu\right) = \frac{q_0^2}{\rho\chi\Theta^2} + \frac{\Delta P_0^2}{3\rho\nu\Theta} \qquad (59)$$

We see that the first order Eckart theory satisfies (53) with $s^\mu = \rho s u^\mu + q_0^\mu/\Theta$, and has the same amount of entropy production as the second order Israel-Stewart theory. However, in this case the entropy current is sourced by q_0 (46) and ΔP_0 (51) which are directly related to gradients of thermodynamic quantities, as opposed to q and ΔP on the right hand side of (53), which are solved for using (45) and (50) respectively. It can also be seen from the above why the viscous stress $\Pi^{\mu\nu}$ does not contribute at first order in the entropy expansion, since $\Pi_{\mu\nu} = \Pi_{0\mu\nu} = -2\rho\nu\langle\nabla_\mu u_\nu\rangle$ produces the right amount of dissipation $\Pi_{0\mu\nu}\Pi_0^{\mu\nu}/(2\rho\nu\Theta)$ on the right hand side of (58), without any corresponding term in the entropy current on the left hand side of (58).

- In the non-relativistic case, we start from (54) and use the classical heat flux $\mathbf{q} = -\rho\chi\boldsymbol{\nabla}\Theta$, the shear stress $\boldsymbol{\Pi} = -2\rho\nu\langle\boldsymbol{\nabla}\mathbf{u}\rangle$, which in component form is explicitly $\Pi_{ik} = -\rho\nu\left(\partial v_i/\partial x_k + \partial v_k/\partial x_i - (2/3)\delta_{ik}\partial v_l/\partial x_l\right)$

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[2] The additional terms when the cross coupling coefficient $c_1 \neq 0$ are of the form $\frac{dq}{d\tau} = ... + (...)\hat{b}^\mu\nabla_\mu\Delta P + (...)\Delta P\nabla_\mu\hat{b}^\mu$ and $\frac{d\Delta P}{d\tau} = ... + (...)\hat{b}^\mu\nabla_\mu q + (...)q\nabla_\mu\hat{b}^\mu$. Therefore, q and ΔP are driven not only by gradients of the background thermodynamic quantities $\hat{b}^\mu\nabla_\mu\Theta$ and $\hat{b}^\mu\hat{b}^\nu\nabla_\mu u_\nu$, but by gradients of each other. Such terms become important in the collisionless limit. The cross coupling coefficients could potentially be used to derive a more accurate model where the dissipative fields are $q_\parallel$, $q_\perp$, $\Delta P_\parallel$ and $\Delta P_\perp$, with the correct coupling between the fields. We leave this to future work.

(Landau & Lifshitz 1987), and the non-relativistic continuity equation to get

$$\frac{\partial(\rho s)}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho s \mathbf{u} + \frac{\mathbf{q}}{\Theta}\right) = \frac{\mathbf{q} \cdot \mathbf{q}}{\rho \chi \Theta^2} + \frac{\boldsymbol{\Pi} : \boldsymbol{\Pi}}{2\rho\nu\Theta} \quad (60)$$

The above is true for the anisotropic case as well, where we have the heat flux $\mathbf{q} = -\rho\chi\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \boldsymbol{\nabla}\Theta$, the shear stress $\boldsymbol{\Pi} = -\Delta P_0 \left[\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \mathbf{I}/3\right]$, and $\Delta P_0 = 3\rho\nu \left[\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\nabla} \cdot \boldsymbol{v}/3\right]$ (Braginskii 1965). Clearly, (60) is the non-relativistic limit of (58), where $s^0 \to \rho s$ when $v/c \ll 1$, since in this limit, $u^0 \to 1$ and $q^0 \to 0$.

Thus, the value of entropy production (right hand side of (53)), which is second order in q^μ and $\Pi^{\mu\nu}$, is the same for the entropy current expanded to first order (relativistic and non-relativistic), as well as to second order. However, expanding the entropy current to third order leads to additional higher order terms on the right hand side of (53) (El et al. 2010).

6. MODEL PARAMETERS: VISCOSITY, CONDUCTIVITY, AND RELAXATION TIME

The parameters of our theory are τ_R, ν, and χ. The entropy production (53) can be interpreted in two ways, depending on whether the plasma is collisional/weakly collisional or collisionless. If the plasma is collisional/weakly collisional, it is the microscopic entropy production due to Coulomb scatterings and τ_R is the Coulomb scattering time scale, i.e., the mean free time between particle-particle collisions. The transport coefficients ν, and χ are set by this time scale τ_R. In particular, both ν and χ are of order $c_s^2\tau_R$ in relativistic collisional kinetic theory:

$$\chi = \phi c_s^2 \tau_R \qquad \text{and} \qquad \nu = \psi c_s^2 h \tau_R \tag{61}$$

where ϕ and ψ are constant dimensionless parameters and $h \equiv 1 + \gamma u/(\rho c^2)$ is the relativistic enthalpy.

For a monoatomic ideal gas in the Chapman-Enskog theory $\phi = (15/4)\psi$. In the non-relativistic Braginskii theory, where Coulomb interactions dominate, $\phi \simeq 4.1\psi$ (see Kulsrud 2005). For a relativistic hard sphere gas (see the clear discussion of Cercignani & Kremer 2001) $\nu \propto T$ for $\Theta \equiv kT/(mc^2) \gg 1$ ($m \equiv$ molecular weight).

In a collisionless plasma, because of the absence of Coulomb scatterings, the scattering time scale diverges $\tau_R \to \infty$ and the entropy does not increase. In this case, (53) is the increase in a *coarse grained* entropy and τ_R is the mean free time between wave-particle scatterings. This is because coarse graining the Vlasov equation leads to a Fokker-Planck equation with an effective collision operator, as is done in quasi-linear theory. Therefore, we still use the closures (61) but with a different interpretation of τ_R compared to the collisional case. Wave-particle scattering is determined primarily by fluctuations in the electromagnetic field that have frequencies of order the cyclotron frequency of the particles of interest (ions, electrons) (Kulsrud 2005). Such fluctuations can either be produced by a cascade from larger scales or by velocity space instabilities that directly excite high frequency fluctuations. Unfortunately, the efficiency of

wave-particle scattering by these processes is not fully understood. Moreover, the high frequency turbulent fluctuations that dominate scattering cannot be resolved in fluid (GRMHD) simulations, so subgrid models of the scattering rate are necessary. The formalism developed in the previous sections is sufficiently general that as the theoretical understanding of wave particle scattering develops, increasingly sophisticated models of τ_R can be implemented in our model. In particular, we stress that the relaxation time, viscosity, and conductivity in our model can be functions of the local plasma conditions (including, e.g., the local plasma β, the amplitude of the turbulent fluctuations, etc.). Here we provide a rough guide to some of the key physics that motivates particular choices for τ_R, ν, and χ.

It is unclear to what extent magnetized turbulence in accretion disks (driven by the magnetorotational instability) produces significant wave-particle scattering. The energetically dominant component of the turbulence (associated with the slow and Alfvén modes) does not produce efficient scattering because it does not cascade to high frequencies (Goldreich & Sridhar 1995; Quataert 1998). The fast mode component can in principle cascade to high frequencies (Yan & Lazarian 2004), but is very strongly damped in low-collisionality plasmas, which likely limits its importance. Absent significant scattering by the high frequency tail of a turbulent cascade, the most important source of scattering is due to velocity space instabilities. These are instabilities in which temperature variations in different directions or local streaming velocities relative to the magnetic field frame provide a source of free energy to drive instabilities (e.g., Stix 1992). Key examples include the firehose, mirror, and ion cyclotron instabilities for the ions and the electron firehose and whistler instabilities for the electrons (we show in §7 that the firehose instability is captured by the fluid model developed in this paper; this is not true for resonant instabilities such as the ion cyclotron and whistler instabilities). The saturation of these instabilities is an active area of research with implications for galaxy cluster plasmas and the solar wind, as well as accretion disks (e.g., Kunz et al. 2014; Riquelme et al. 2015; Sironi & Narayan 2015; Hellinger et al. 2015). Note that for the ions in RIAFs, a non-relativistic theory ($\theta_i \equiv kT_i/(m_ic^2) \lesssim 1$) is sufficient, since $\theta_i \sim (H/r)^2(GM/(rc^2))^{1/2}$, and $r > H \equiv$ scale height. However, for the electrons a relativistic theory is required since in the RIAF model $\theta_e \gtrsim 1$ and indeed relativistic electrons are observed close to Sgr A* (Doeleman et al. 2008).

Velocity space instabilities in low collisionality plasmas such as RIAFs are believed to arise because shearing, heating, expansion, or compression drives the plasma towards an unstable configuration, initiating the instability. The scattering rate produced once the velocity space instabilities saturate is of the order of the timescale on which the instabilities are driven (see Kunz et al. 2014; Riquelme et al. 2015). This balance between driving and scattering timescales maintains the plasma near the marginal state for the instability. These considerations provide good motivation for one specific choice of the relaxation time in low-collisionalty plasmas: τ_R of the order of the dynamical time τ_d, because the lat-

ter sets the characteristic timescale for heating, expansion, etc. For black hole accretion problems, this suggests $\tau_R \sim \tau_d \simeq (GM/r^3)^{-1/2}$.

More specifically, however, velocity space instabilities will only set in if the free energy driving the instability is sufficiently large. For example, the heat flux due to conduction cannot exceed the saturated value of $q \simeq \Phi\rho c_s^3$, where Φ is a dimensionless constant of order unity, but the electron whistler instability implies an even more stringent limit on the heat flux for high β plasmas (Pistinner & Eichler 1998). As a second example, a non-relativistic plasma is firehose (mirror) unstable if $\Delta P/P \leq -2/\beta$ ($\Delta P/P \gtrsim 1/\beta$). Observations of the solar wind show that the plasma pressure anisotropy obeys these constraints to reasonable accuracy (e.g., Hellinger et al. 2006; Bale et al. 2009). Thus a physical model of accretion disk viscosity should ensure that $|\Delta P|$ does not exceed these bounds. Since, both viscosity and thermal conductivity are related to the relaxation time scale τ_R in (61), this implies a modification of the thermal conductivity as well, when τ_R is decreased.

One strategy for reducing τ_R in the presence of a small scale instability is to set

$$\tau_R = \tau_d f\left(\frac{x}{x_{crit}}\right) \tag{62}$$

where x is some parameter (e.g. ΔP) that has a critical value x_{crit} for instability. The function f is arbitrary but should have (1) $f(0) = 1$; (2) $f'(0) = 0$; (3) $f(1) \ll 1$. One function with the desired properties for f is the Fermi-Dirac distribution:

$$f(y) = \frac{1}{e^{(y-1)/\lambda} + 1} \tag{63}$$

for $y > 0$, and $f(y) = 1$ for $y < 0$. λ is an adjustable parameter that determines the width of the transition to small τ_R. As an example, saturated conduction can be implemented by sending $\tau_R \to \tau_d f(q/q_{crit})$, where $q_{crit} = \Phi\rho c_s^3$ is the maximum heat flux (and we have assumed a characteristic, unsaturated, relaxation time of τ_d)[3]. To consider several instabilities with instability threshold ratios $y_1, y_2, y_3...$, one can set $\tau_R = \tau_d f(y_1) f(y_2) f(y_3)...$

To understand the effect of (62) and the associated reduction in the relaxation time near an instability threshold, consider an instability with threshold pressure anisotropy ΔP_{crit}. As $\Delta P \to \Delta P_{crit}$, the relaxation time is reduced. Because $\Delta P_0 \propto \nu \propto \tau_R$, this reduces both ΔP_0 and the time required for ΔP to relax to ΔP_0. Thus the plasma quickly falls below the instability threshold. Once below the threshold, τ_R becomes τ_d again, and ΔP relaxes to the large ΔP_0 over a dynamical time scale. When the threshold is crossed again, (62) takes effect and the cycle restarts. The net effect of this cycle is to

set up a feedback loop that results in q and ΔP being pinned to their values at an instability threshold, in a statistical sense. At every instant, the process generates entropy according to (53), which only involves q and ΔP and not q_0 and ΔP_0.

7. LINEAR THEORY

Here we address the following questions: (1) what are the characteristic speeds in our extended MHD model? These are needed to determine the Courant condition in explicit numerical evolution; (2) which choices of model parameters τ_R, χ, ν yield a stable model when the initial state is an equilibrium? (3) under what conditions does one recover the firehose instability of an initially anisotropic state? (4) how large a heat flux can the model admit before it loses its stability and hyperbolicity? (5) what is the stability of the model in a frame not comoving with the fluid? and finally (6) under what conditions is the model causal?

Consider an initially homogeneous magnetized fluid in Minkowski space. The initial state in the fluid frame has $u^\mu = \{1, 0, 0, 0\}$, $\rho = \rho_0$, $u = u_0$, $b^\mu = (0, b\sin\theta, 0, b\cos\theta)$, and $q = \Delta P = 0$. The initial temperature is is $T_0 = P_0/\rho_0 = (\gamma - 1)u_0/\rho_0$. We perturb around this initial state, e.g. $q \to 0 + \delta q$ with $\delta q \propto \exp(ikx - i\omega t)$, linearize, and find the dispersion relation $D(\omega, k) = 0$.

It is worth first revisiting the linear theory for relativistic ideal MHD. As usual, the Alfvén waves factor and the Alfvén velocity is

$$v_A^2 = \frac{b^2}{\rho_0 h_0 + b^2}. \tag{64}$$

The slow and fast modes combine in a complicated, fourth-order dispersion relation. The special cases parallel and perpendicular to the field give the sound speed

$$c_s^2 = \frac{\gamma P_0}{\rho_0 h_0} \tag{65}$$

and the fast magnetosonic speed

$$v_M^2 = \frac{b^2 + \gamma P_0}{\rho_0 h_0 + b^2} \tag{66}$$

respectively.

A rigorous stability analysis would require a study of the general, ninth-order dispersion relation. Instead we consider the special cases of propagation parallel and perpendicular to the magnetic field lines for conduction only and for viscosity only, and analyze the resulting dispersion relations.

$\nu = 0$, $k \parallel B$. The four Alfvén modes are unaffected, as one might expect because they do not perturb the temperature. The remaining four modes arise from the entropy mode and the sound waves coupled to evolution of δq. The dispersion relation is:

$$(\omega^2 - c_s^2 k^2)\omega(\omega\tau_R + i) - \frac{\chi c_s^2}{\gamma c^4}\left((\omega^2 - (\gamma - 1)c^2 k^2)^2 + c^2 k^2(\omega^2 - c_s^2 k^2)\frac{\gamma(\gamma - 1)c^2}{c_s^2}\right) = 0. \tag{67}$$

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[3] Saturated conduction is technically a collisionless effect and in this limit, the anisotropic pressure depends not only on the shear

projected onto the field lines, but also on the gradients of the heat flux which we have ignored. See also footnote 2.

Evidently, the sound waves emerge in the ideal limit.

$\nu = 0$, $k \perp B$. We recover five zero frequency modes, two magnetosonic modes, and one new mode:

$$\omega = -i\left(\tau_R - \frac{\chi c_s^2}{\gamma c^4}\right)^{-1}. \tag{68}$$

$\chi = 0$, $k \parallel B$. We recover four Alfvénic modes (propagating in each direction with two polarizations), a zero frequency mode that asymptotes to the entropy mode, and modes that couple sound waves and ΔP:

$$\omega^3 + i\omega^2\frac{1}{\tau_R} - \omega\left(\frac{4\nu k^2}{3\tau_R(1 + \gamma u_0/\rho_0 c^2)} + c_s^2 k^2\right) - i\frac{c_s^2 k^2}{\tau_R} = 0. \tag{69}$$

$\chi = 0$, $k \perp B$. We recover a damped viscous mode coupled to the two magnetosonic modes:

$$\omega^3 + i\omega^2\frac{1}{\tau_R} - \omega\left(\frac{\nu k^2 \rho_0 c^2}{3\tau_R(b^2 + \rho_0 c^2 + \gamma u_0)} + k^2 v_M^2\right) - i\frac{k^2 v_M^2}{\tau_R} = 0. \tag{70}$$

7.1. Characteristic Speeds

The introduction of evolution equations for q and ΔP makes the equations hyperbolic and therefore also introduces new characteristic speeds. On dimensional grounds we expect a characteristic speed associated with conduction $v_q^2 \propto \chi/\tau_R$ and a characteristic speed associated with viscosity $v_{\Delta P}^2 \propto \nu/\tau_R$. What are the constants of proportionality?

First consider the pure viscosity case in the limit $k \to \infty$. For parallel propagation

$$\omega^2 = k^2\left(c_s^2 + \frac{4}{3}\frac{\nu}{h_0\tau_R}\right) \tag{71}$$

and for perpendicular propagation

$$\omega^2 = k^2\left(v_M^2 + \frac{1}{3}\frac{\nu}{\tau_R(1 + \gamma u_0/\rho_0 + b^2/\rho_0)}\right). \tag{72}$$

This motivates the definition

$$v_{\Delta P}^2 \equiv \frac{4}{3}\frac{\nu}{h_0\tau_R} \tag{73}$$

as a characteristic viscous speed. In our closure model $v_{\Delta P} = c_s\sqrt{4\psi/3}$, and in a numerical implementation the Courant condition needs to be adjusted accordingly.

Next consider the pure conduction case in the limit $k \to \infty$. For parallel propagation the general solution for ω^2 for the coupled entropy-conduction-sound wave is complicated and not much simpler than the dispersion relation itself, but the expression

$$\omega^2 = \frac{1}{2}k^2\left(c_s^2 + v_q^2 \pm (c_s^4 + v_q^4)^{1/2}\right) \tag{74}$$

provides a reasonable first approximation (although it does not reveal that $\omega^2 \propto 1/(\gamma(\gamma-1)c^4 - c_s^2 v_q^2)$, which leads to slightly higher signal speeds when $c_s, v_q \sim c$). Here

$$v_q^2 \equiv (\gamma - 1)\frac{\chi}{\tau_R} \tag{75}$$

is a characteristic conduction speed. For perpendicular propagation, the excitation of q is damped and nonpropagating. In our closure model $v_q = c_s\sqrt{\phi(\gamma-1)}$ and so

in a numerical implementation of the model the Courant condition must be adjusted accordingly.

7.2. Stability

It is well known that the Eckart first order conduction model (see Misner et al. 1973, p. 567) is subject to a dramatic instability with growth rate $\propto 1/\chi$ (Hiscock & Lindblom 1985). The Eckart theory is recovered in the limit $\tau_R \to 0$ in our model, so we expect that it will be unstable if we set τ_R too small.

To make this idea more precise, consider the $k \to 0$ limit of (67). We find

$$\omega = -i\left(\tau_R - \frac{\chi c_s^2}{\gamma c^4}\right)^{-1} \tag{76}$$

where we have temporarily restored factors of c and $c_s^2 = \gamma P_0/(\rho_0 + \gamma u_0/c^2)$. Evidently the model is unstable if

$$\tau_R < \chi c_s^2/(\gamma c^4) \tag{77}$$

which is the same instability condition found by Hiscock & Lindblom (1985) for an isotropic viscosity. For propagation perpendicular to the field, (68), we recover the same stability condition. Setting $\chi = \phi c_s^2 \tau_R$, instability requires $\phi > \gamma c^4/c_s^4$ for parallel or perpendicular propagation.

In the comoving frame, the viscous modes are stable if $\Delta P = 0$ in the initial conditions. All parallel and perpendicular modes are either neutral or damped. The stability for viscous modes in a non-comoving frame will be discussed in 7.5.

7.3. Firehose Instability

In non-relativistic Braginskii theory, the inclusion of anisotropic conduction and viscosity opens up avenues for qualitatively new types of linear instabilities. For example, in the presence of anisotropic conduction, there are buoyancy instabilities driven by temperature gradients rather than entropy gradients (Balbus 2000; Quataert 2008). In the presence of anisotropic viscosity, the character of the magnetorotational instability can change, with viscous transport of angular momentum driving an instability even when magnetic tension is negligible (Quataert et al. 2002; Balbus 2004). And, finally, a background pressure anisotropy can itself be subject to a host of instabilities, at least one of which (the firehose instability) is well captured by the EMHD-type models described here (e.g., Schekochihin et al. 2009). We expect that much of this rich physics will carry over to the relativistic theory described in this paper since it reduces to Braginskii theory in the non-relativistic limit. As a simple concrete example of this, we derive here the relativistic instability criterion for the firehose instability.

Pressure anisotropy modifies the propagation speed of Alfvén waves and, if ΔP is large enough, transforms them into unstable, non-propagating modes. Firehose instability has not appeared until now because our equilibrium has $\Delta P = 0$. But we can recover the firehose instability by setting $\Delta P \neq 0$ in the background state and taking $\tau_R \to \infty$ so that the pressure anisotropy cannot decay and the initial state is an equilibrium.

The resulting dispersion relation for parallel propagation is:

$$\omega^2 = k^2 \frac{b^2 + \Delta P}{b^2 + \rho_0 + \gamma u_0 + \frac{1}{3}\Delta P}. \tag{78}$$

For $\Delta P = 0$ we recover the Alfvén waves. For $\Delta P > 0$ the propagation speed increases and becomes superluminal if $\Delta P > \frac{3}{2}(\rho_0 + \gamma u_0)$. However, this would require $\Delta P / P > 3\gamma / (2(\gamma - 1))$. For $\Delta P < 0$ we recover the relativistic firehose instability criterion:

$$b^2 + \Delta P < 0 \;\Rightarrow\; \text{INSTABILITY}. \tag{79}$$

This criterion is consistent with the relativistic kinetic criterion of Lerche (1966).

7.4. *Instability for Large q/u*

Hiscock & Lindblom (1988) (hereafter HL88) have shown that the Israel-Stewart theory loses stability and hyperbolicity above a critical value of $q/u \simeq 0.08898$ in the ultrarelativistic ($u \gg \rho$) limit. What does this imply for the EMHD model?

Consider an initial state with a background heat flux $q = q_0$. We will perturb around this state to test stability. But to do this, we are obliged to take the limit $\tau_R \to \infty$ so that the initial state with $q = q_0$ becomes an equilibrium. It matters how this limit is taken: if we assume $\chi \propto \tau_R$, terms proportional to both τ_R and χ must be retained. Linearization around this equilibrium for modes with $k \parallel B$ yields a fourth-order dispersion relation for the coupled sound/entropy/conduction excitations.

The stability of the $q = q_0$ equilibrium depends on γ, ϕ, q, ρ, and u in a complicated way. To compare with HL88 we consider the ultrarelativistic limit with $\gamma = 4/3$ and $u \gg \rho$; HL88 also make a choice for the coefficient b_1 that is equivalent to setting $\phi = 12/5$. This simplifies the analysis, and one can show that the discriminant of

the dispersion relation (which in this limit has real coefficients) changes sign at $q_0/u = 0.08898$, as in HL88. Indeed, one can show that HL88's $\lambda = 1$ Israel-Stewart model is identical to ours if we restrict attention to motion along the field.

We can generalize HL88's analysis by allowing ϕ to be different from 12/5. The critical $q_0/u = 2^{1/2}/3$ as $\phi \to 0$. Between $\phi = 0$ and $\phi = 3$ the critical q_0/u is slightly above $q_0/u = (3 - \phi)/10$, still in the ultrarelativistic limit.

Is the instability a consequence of the second-order terms in the evolution equation for q? The higher order terms enter the linear theory only when $q_0 \neq 0$, so they have played no role until now. Turning these terms off and repeating the stability analysis, one finds the discriminant of the dispersion relation is positive definite. The instability is therefore seated in the second order terms. This is relevant for numerical implementations of the EMHD model, since it suggests that manipulating the higher order terms may improve stability, albeit at the cost of violating the second law.

7.5. *Stability in a non-comoving frame*

It is an interesting feature of the Israel-Stewart theory that in some cases perturbations comoving with the fluid appear stable, but even a small relative velocity between the frame in which the perturbation is defined and the fluid, makes the perturbation unstable (Hiscock & Lindblom 1985). Consider a background velocity $u^\mu = (\Gamma, u^x, 0, 0)$, with $\Gamma = \sqrt{1 + u_x^2}$ the Lorentz factor, a background magnetic field $b_\mu = (-b_x u^x/\Gamma, b_x, 0, 0)$, and perturbations $(\delta\rho, \delta u_x, \delta\Delta P)$ of the density, longitudinal velocity and pressure anisotropy. From the linearly perturbed equations of mass conservation, evolution of the pressure anisotropy, and $\nabla_\mu \delta T_x^\mu = 0$, we get (in the absence of heat conduction)

$$\Gamma \partial_t(\delta\rho) = \rho_0 u_x \partial_t(\delta u_x), \tag{80}$$

$$3\rho_0 h_0(1 + 2u_x^2)\partial_t(\delta u_x) = 2u_x \Gamma^2 \partial_t(\delta\Delta P) - 3u_x \Gamma^2 \partial_t(\delta\rho), \tag{81}$$

$$2\rho_0 \nu b_2 \Gamma^2 \partial_t(\delta\Delta P) = -\Gamma\delta\Delta P + 2\rho_0 \nu u_x \partial_t \delta u_x. \tag{82}$$

Substituting for $\partial_t \delta u_x$ and $\partial_t \delta\rho$, we get the evolution equation for $\delta\Delta P$,

$$\partial_t(\delta\Delta P) = -\frac{3(2\rho_0(h_0 - 1)u_x^2 + \rho_0 h_0)}{3b_2(\rho_0 h_0 \Gamma^2 + \rho_0(h_0 - 1)u_x^2) - 2u_x^2} \frac{\delta\Delta P}{2\rho_0 \nu \Gamma}. \tag{83}$$

In this case, the stability condition is

$$b_2 > \frac{2u_x^2}{3(\rho_0 h_0 \Gamma^2 + \rho_0(h_0 - 1)u_x^2)}. \tag{84}$$

Note that b_2 is the coefficient of the second order contribution to the entropy current (35) due to shear stress and is equal to $\tau_R/(2\rho_0 \nu)$.

7.6. *Causality*

For small τ_R, i.e., for small b_1 and b_2, the model is no longer causal. To demonstrate this, we consider the prin-

cipal part of the evolution equations at a point at which $u_\mu = (-1, 0, 0, 0)$, and $b_\mu = (0, b_x, 0, 0)$, and compute the characteristic speeds of the system in the tetrad frame and along the direction $\overrightarrow{e}_{(x)}$. Writing the linearized system of equations as

$$A^t \partial_t(\delta U) + A^i \partial_i(\delta U) + B\delta U \tag{85}$$

for perturbations δU, with matrices A^μ, B depending on the unperturbed variables U_0, then the characteristic speeds v along direction i are solutions of the equation

$$det(v * A^t - A^i) = 0. \tag{86}$$

In the absence of heat conduction, the relevant parts of the equations for $(\rho, u, u_x, \Delta P)$ are then

$$\partial_t \rho = -\rho \partial_x u_x + \ ... \tag{87}$$

$$\partial_t u = -(u + P)\partial_x u_x + ... \tag{88}$$

$$\rho h \partial_t u_x = -\partial_x P + \frac{2}{3}\partial_x \Delta P + \ ... \tag{89}$$

$$\partial_t \Delta P = \frac{1}{b_2}\partial_x u_x + \ ... \tag{90}$$

(the transverse components of the magnetic field and velocity still follow the standard dispersion relation for Alfven waves). The characteristic speeds of the system then depend on the equation of state $P(\rho, u)$. For a simple gamma-law fluid, $P = (\gamma - 1)u$, we find the speeds $(0, 0, \pm\sqrt{(2\rho\nu + 3(\gamma-1)\gamma u b_2)/(3\rho h b_2)})$. For small values of b_2, two of the speeds are thus greater than $c = 1$. To avoid superluminal speeds, we need

$$b_2 > \frac{2}{3(\rho + \gamma u(2 - \gamma))}. \tag{91}$$

Rewriting in terms of the relaxation time scale τ_R, the condition for causality is

$$\tau_R > \frac{4\rho\nu}{3(\rho + \gamma u(2 - \gamma))}. \tag{92}$$

This is identical to what we would have inferred from equation (71) by imposing $\omega/k < c$. Regardless of our choice of frame, the theory is thus problematic for small b_2, i.e. small τ_R or large ν/τ_R. These results are once more reminiscent of what Hiscock & Lindblom (1983) found for an isotropic viscosity. In that case, they derived the stability condition $b_2 > 2/(3\rho h)$. Hiscock & Lindblom (1983, 1988b) also demonstrated that the stability of the theory implied that the system of evolution equations was hyperbolic and causal, and that if the system of equations was either not causal or not hyperbolic, it was unstable.

8. NONLINEAR THEORY

In this section we give a brief but incomplete description of the behavior of the model in shocks.

Ideal hydrodynamics allows shocks and contact (entropy) discontinuities. The boundary conditions at the discontinuity are determined by continuity conditions on the fluxes of mass, momenta, and energy.

Once viscosity is included in a non-relativistic fluid model the discontinuity is replaced by a sharp but continuous transition of width Δx such that the diffusion timescale $\Delta x^2/\nu$ is comparable to the transit time through the shock $\Delta x/v$, so $\Delta x \sim \nu/v$. If $\nu \sim \lambda_{mfp}c_s$ and $v \sim c_s$ then $\Delta x \sim \lambda_{mfp}$. Put differently, viscosity can propagate information upstream at characteristic speed $\sim \nu/\Delta x$, which is supersonic if the structure is narrow enough. Conduction produces an upstream precursor but does not remove the discontinuity (Mihalas & Weibel Mihalas 1984).

In Maxwell-Cattaneo type theories there is a new characteristic speed, $\propto (\nu/\tau_R)^{1/2}$; if $\nu \sim c_s^2\tau_R$ then this speed is comparable to c_s. If the upstream velocity exceeds this, the shock is still discontinuous. In Israel-Stewart theory, this was studied by Olson & Hiscock (1990). See

Bouras et al. (2010) for a numerical study of shocks in this theory and a comparison to shock solutions obtained using kinetic theory.

Linear analysis showed (§7) that our EMHD model contains a speed $\sim (\chi/\tau_R)^{1/2} \sim \phi^{1/2}c_s$ associated with thermal conduction and another speed $\sim \psi^{1/2}c_s$ associated with viscosity. One would then expect the model to have a discontinuity over a sufficiently strong shock. How, then, do ΔP and q change across the discontinuity?

The variation of ΔP and q depend on the shock substructure, essentially because there are no continuity conditions, such as those present in ideal MHD, that can be used to navigate across the shock. To clarify this point, consider the conduction model governed by equation (47) in the frame of a strong (so that v upstream is greater than v_q) steady shock at $x = 0$, with $u^x > 0$. The dominant terms near the shock are

$$u^x \partial_x Q = \frac{1}{\tau_R}\left(\frac{\tau_R}{\rho\chi P}\right)^{1/2}\left(-\rho\chi\hat{b}^x(\partial_x\Theta + \Theta u^x \partial_x u_x)\right) \tag{93}$$

because both $\partial_x T$ and $\partial_x u_x$ are large inside the shock. This equation has the form

$$\partial_x Q = F(\rho, \Theta, u^x)\delta(x) \tag{94}$$

where δ arises from ∂_x acting on the discontinuities. If F were continuous through the shock we could integrate this equation across the jump and find a definite solution for Q at $x > 0$. But F is changing discontinuously through the shock and hence the postshock Q depends on the substructure of the shock.

EMHD shocks contain substructure on three scales.

(1) On small scales the width of the shock is set by bulk viscosity.

Because bulk viscosity is not included in the model, this smallest scale is unresolved and the evolution of q and ΔP across the shock is undetermined (but the change in, e.g., Q across the shock $\equiv \Delta Q$ is limited, since if F in equation (94) is monotonic then ΔQ must lie between F in the downstream state and F in the upstream state). In a numerical evolution it will be set by the numerical closure.

(2) On intermediate scales the pressure anisotropy and heat flux relaxation produce a tail of width $\sim u^x\tau_R$.

The separation of scales (1) and (2) is an artifact of the model since if τ_R is small inside the shock, (1) and (2) can be comparable.

(3) Far from the shock ΔP and $q \to 0$ and the shock obeys the ideal MHD jump conditions.

The latter point implies that unless one is interested in shock substructure (and it would seem that EMHD is not the ideal model for collisionless shocks), the EMHD model may provide an adequate description of flow more than a few mean free paths from the shock.

9. DISCUSSION AND CONCLUSION

In this paper, we have derived an extended MHD model for a relativistic weakly collisional plasma incorporating the effects of anisotropic conduction and viscosity. We are motivated by applications to low accretion rate black holes, but the model can also be applied in other contexts such as neutron star magnetospheres. This is the first in a series of publications. While this

paper deals with the derivation, linear theory and stability analysis of our model, a later publication will discuss a flexible new code which we have written to solve this model numerically, as well as possible future extensions of this model. As tests for the code, and to get insights into the model, we derive various analytic/semi-analytic solutions to this model. The code, as well as the various tests will be described in Chandra et al. (2015). Finally, we shall apply this model and the code to study the dynamics and structure of an accretion disk around a slowly accreting Kerr black hole.

9.1. *Summary of the Governing Equations*

The complete model is given by the usual continuity equation (4), energy-momentum equations (7) and induction equation (15), supplemented by a heat conduction model (25) and shear viscosity model (28), together with evolution equations for the magnitude of the heat flux q (45) and the pressure anisotropy ΔP (50).

9.2. *Summary of the Formalism*

We deduced the form of the heat flux (25) and the viscous stress (28) by examining the symmetries of the system in the presence of a magnetic field and by appealing to the small values of the gyroradius and the gyroperiod in astrophysical plasmas. The evolution equations (45) and (50) are then derived using the Israel-Stewart theory of dissipative hydrodynamics. We have expanded the entropy current up to second order in deviations from equilibrium and then enforced its divergence to be positive. From this simple principle, we get equations for the heat flux and the pressure anisotropy in a magnetized plasma. The derivation naturally shows that, as in non-relativistic plasmas, 1) the heat flux is driven by temperature gradients projected along the field lines and that 2) the pressure anisotropy is driven by a background shear projected along the field lines. Since our model is based on the Israel-Stewart theory, it conditionally satisfies all the requirements of dissipative theories in general relativity: hyperbolicity, causality and stability. For example, in the ultrarelativistic limit ($u \gg \rho$), the model is hyperbolic as long as $q/u < 0.089$.

In the limit where the relaxation time scale $\tau_R \to 0$, our model reduces to that of Braginskii (1965). The hyperbolic equations for parallel heat flux and pressure anisotropy relax to forms (46) and (51) respectively, which are covariant generalizations of the Braginskii closure. The pressure anisotropy in Braginskii theory arises due to a balance between the conservation of adiabatic invariants and pitch-angle scattering. In the appendix, we show using relativistic kinetic theory that this is true of our model as well. In the non-relativistic limit, the inclusion of anisotropic conduction leads to many new types of instabilities, such as the magnetothermal instability (Balbus 2000) and the heat flux driven buoyancy instability (Quataert 2008), both of which are also modified by the inclusion of anisotropic viscosity (Kunz 2011). Our relativistic model should also display all the instabilities that arise due to inclusion of anisotropic dissipative effects. As an example, we showed that our model reproduces the correct threshold for the firehose instability.

We relate the transport coefficients in our model (χ, ν) to the effective scattering time scale τ_R. A collisionless plasma is subject to a number of kinetic instabilities

(mirror, firehose, ion cyclotron, electron whistler) that effectively regulate the pressure anisotropy and the heat flux. Our closure prescription incorporates the isotropizing effects of these instabilities through the modification of τ_R, thus providing a convenient way to incorporate the various kinetic effects. The exact way in which τ_R should be modified will ultimately be answered by first principle particle-in-cell calculations, which are well under way (Kunz et al. 2014; Riquelme et al. 2015).

9.3. *Two-Fluid Effects and Observational Signatures*

The model is a one-fluid model of a plasma consisting of multiple species, here electrons and ions. The one-fluid matter stress tensor $T^{\mu\nu}_{\text{matter}}$ in (21) has been obtained by summing up stress tensors of each individual species in (3). Therefore, the rest mass density ρ, the internal energy u, the pressure P, the heat flux q^μ and the shear stress $\Pi^{\mu\nu}$ in (21) are all a sum of the respective quantities of both electrons and ions (using $u^\mu_e \approx u^\mu_i \equiv u^\mu$), i.e. $\rho = \rho_e + \rho_i$, $u = u_e + u_i$, $P = P_e + P_i$, $q^\mu = q^\mu_e + q^\mu_i$ and $\Pi^{\mu\nu} = \Pi^{\mu\nu}_e + \Pi^{\mu\nu}_i$. The temperature Θ in the model is a combination of both electron Θ_e and ion temperatures Θ_i, i.e., $\Theta = (m_i\Theta_i + m_e\Theta_e)/(m_i + m_e)$ (since $P = P_e + P_i$), where m_e and m_i are the electron and ion masses, $\Theta_e \equiv kT_e/(m_ec^2)$, and $\Theta_i \equiv kT_i/(m_ic^2)$. Introduction of a single heat flux q^μ and a single shear stress $\Pi^{\mu\nu}$ for a system with multiple species is inconsistent, unless we make further approximations.

The ratio of the relaxed electron to ion heat flux is $q_{e0}/q_{i0} \sim (P_e/P_i)^2(\tau_e/\tau_i)(m_i/m_e)(L_i/L_e)$, where L_e and L_i are the characteristic scales for the variation of the electron and ion temperatures and τ_e and τ_i are the electron and ion relaxation time scales respectively. Assuming $L_e \sim L_i$ and $\tau_e \sim \tau_i$, our one-fluid model is a reasonable description of the overall dynamics when $P_e/P_i \ll \sqrt{m_e/m_i}$. In this limit, the contribution to the total heat flux is dominated by the ion heat flux $q^\mu \approx q^\mu_i$, and the single temperature Θ in our one-fluid model is the ion temperature $\Theta \approx \Theta_i$. Similarly, the ratio of the relaxed electron to ion pressure anisotropy is $\Delta P_{e0}/\Delta P_{i0} \sim (P_e/P_i)(\tau_e/\tau_i) \ll 1$, when $P_e/P_i \ll 1$ (again assuming $\tau_e \sim \tau_i$). In this limit, the total pressure anisotropy is dominated by the ions and thus we have for the total shear stress $\Pi^{\mu\nu} \approx \Pi^{\mu\nu}_i$. Note that in the collisional (Braginskii) regime, $P_e \simeq P_i$, and $\tau_e/\tau_i \sim \sqrt{m_e/m_i}$, where τ_e and τ_i are the Coulomb scattering time scales for electrons and ions. The dominant heat flux is now due to electrons $q_{e0}/q_{i0} \sim \sqrt{m_i/m_e} \gg 1$, while the dominant pressure anisotropy is still due to ions $\Delta P_{e0}/\Delta P_{i0} \sim \sqrt{m_e/m_i} \ll 1$. Since in this collisional limit, $T_e \simeq T_i$, evolving a single temperature $\Theta \approx 2\Theta_i$, as we do in this paper is appropriate.

Generalizing this one-fluid model to a two-fluid model that accounts for both electron and ion dissipation can be done in a number of ways. One approach is to use separate conservation equations for both the electrons and ions, into which our model can be incorporated in a straightforward manner. This however includes kinetic length scales in the system, which cannot be resolved in a global accretion disk simulation. Another approach is to work within the MHD ordering, which only requires one additional variable, the electron temperature. Ressler et al. (2015) have developed such a model by us-

ing a separate electron entropy equation into which they incorporate electron conduction using a reduced version of the anisotropic conduction equation that we have derived here. The emission from the plasma is dominated by the electrons and hence the electron thermodynamical quantities are crucial to predict observables such as spectra, images and light curves.

The non-ideal effects modeled here may have a number of implications for the dynamics and observational properties of slowly accreting black holes: 1) conduction can redistribute energy spatially, changing where the emission comes from and potentially changing the dynamics by increasing the pressure at large latitude/radii and 2) viscosity can provide an additional source of transport and heating (over and above the Maxwell and Reynolds stress), potentially modifying $\dot{M}$ and the thermodynamics of the plasma – the latter being relevant for the emission.

9.4. *Connection to Other Models*

The model we have presented is the simplest in a hierarchy of possible models incorporating non-ideal effects in magnetized plasmas. In particular, it only includes a single parallel heat flux $q^\mu = q\,\hat{b}^\mu$ and the parallel and perpendicular pressures are controlled by a single variable, the pressure anisotropy ΔP. The derivation was based on thermodynamic principles. On the other hand, models of collisionless plasmas are usually derived by taking velocity space moments of the Vlasov equation. Early examples of relativistic models derived in this way are by Gedalin (1991), Tsikarishvili et al. (1992) and Tsikarishvili et al. (1994). They have independent variables for the parallel ($P_\parallel$) and perpendicular ($P_\perp$) pressures. However, their stress tensors lack a heat flux and their models reduce in the non-relativistic limit to the well-known CGL closure (Chew et al. 1956). A more recent model by TenBarge et al. (2008) has evolution equations for $P_\parallel$ and $P_\perp$ and these couple individually to separate heat fluxes $q_\parallel \sim \hat{b}^\mu \nabla_\mu P_\parallel$ and $q_\perp \sim \hat{b}^\mu \nabla_\mu P_\perp$. In the non-relativistic regime, when the mean free path is small compared to the system size, such a model recovers the Braginskii limit (Snyder et al. 1997). Thus, we expect our model to be formally applicable only in this regime. Also, since we have not derived our model using kinetic theory, we do not expect that it will reproduce all of the kinetic linear modes; for example, our model lacks the mirror instability. However, a nice feature of our model is that it includes collisions whereas the previous relativistic models for collisionless plasmas do not. To include collisions in the moment formalism that the previous models used, one has to include a collision operator in the Vlasov equation and take its moments. Our model naturally includes collisions because it is an expansion up to second order around thermal equilibrium. The relaxation time scale τ_R can be interpreted as the relaxation time scale in the BGK collision operator. Its presence is convenient since it makes it easy to incorporate subgrid models of the saturation of kinetic plasma instabilities as an effective collisionality.

Our model captures the leading order effects of heat transport and pressure anisotropy while still being relatively simple. More sophisticated models have been derived from Israel-Stewart theory in the presence of a magnetic field (Huang et al. 2010), albeit in a different context, for use in strange quark stars with strong magnetic fields. The Israel-Stewart theory itself is a specific instance of a class of theories derived from Extended Thermodynamics, which are based on an entropy principle, such as the one we have used in this paper. See Jou et al. (1988) for a review. While this formalism gives us the evolution equations for the dissipative fluxes, one needs to eventually resort to kinetic theory in order to compute the transport coefficients. To perform this computation, Israel & Stewart (1979) have used an approach similar to that of Grad (1949). They expand the distribution function in momentum space polynomials around an equilibrium, i.e., $f = f_0(1 + a_\mu p^\mu + b_{\mu\nu} p^\mu p^\nu)$, where f_0 is the equilibrium distribution function. This ansatz for the distribution function is then used along with the second moment of Boltzmann equation to compute the transport coefficients, which come out in terms of moments of f_0. Note that the second moment of the Boltzmann equation is one moment higher than the divergence of the stress-tensor and, it is at this level of the moment hierarchy where the collision operator makes a non-zero contribution.

APPENDIX

ADIABATIC INVARIANTS AND PRESSURE ANISOTROPY

Equation 51 in the main text for ΔP_0 is the natural general relativistic generalization of the well-known Braginskii (1965) relation between pressure anisotropy, viscosity, and shear stress in a non-relativistic plasma. In the non-relativistic limit, this relationship also has a simple interpretation in terms of a balance between (1) the rate of generation of pressure anisotropy by adiabatic invariance of the magnetic moment in a slowly varying magnetic field and (2) the isotropization in velocity space by pitch-angle angle scattering at a rate ν_s. We briefly review this non-relativistic result and then discuss its generalization to relativistic kinetic theory of a magnetized plasma. This provides a useful physical interpretation of the thermodynamic derivation of the equilibrium pressure anisotropy (eq. 51) in

the main text. We take $k = m = c = 1$ throughout this Appendix.

Non-relativistic Kinetic Theory

The viscous stress tensor in a magnetized non-relativistic collisional plasma in which the cyclotron frequency is much larger than the collision frequency is given by (Braginskii 1965)

$$\mathbf{\Pi} = -3\rho\nu \left[\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \boldsymbol{\nabla}\boldsymbol{v} - \frac{\boldsymbol{\nabla}\cdot\boldsymbol{v}}{3} \right] \left[\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathbf{I}}{3} \right], \tag{A1}$$

Equation A1 can also be written as

$$\mathbf{\Pi} = -\Delta P \left[\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{\mathbf{I}}{3} \right], \tag{A2}$$

where the pressure anisotropy in the non-relativistic limit is given by

$$\Delta P = 3\rho\nu \left[\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \boldsymbol{\nabla}\boldsymbol{v} - \frac{\boldsymbol{\nabla}\cdot\boldsymbol{v}}{3} \right]. \tag{A3}$$

ΔP in equations A2 and A3 is defined as $\Delta P = P_\perp - P_\parallel$, with directions defined by the local magnetic field in the plasma. Note that since $\nu \sim c_s^2/\omega$ where ω is the pitch angle scattering rate and c_s is the sound speed, equation A3 is equivalent to

$$\omega \frac{\Delta T}{T} \sim \frac{d}{dt} \ln \left[\frac{B^3}{\rho^2} \right] \tag{A4}$$

where we have used the induction equation and mass conservation to rewrite the right-hand-side of equation A4. The left hand side of equation A4 is the rate at which scattering at rate ω decreases the pressure anisotropy. The right hand side of equation A4 is the rate at which adiabatic invariance of $T_\perp/B$ (magnetic moment) and $T_\parallel B^2/\rho^2$ (the 'bounce' invariant) change the pressure anisotropy. The assumption of collisional theory is that these two effects approximately balance each other.

Relativistic Kinetic Theory

The thermodynamic derivation in §4 (in particular eq. 51) shows that results very similar to equations A3 and A4 relate pressure anisotropy and viscosity in GR. To understand the microscopic origin of these results, it is instructive to consider the relativistically correct first and second adiabatic invariants for an individual particle (Sturrock 1994):

$$\frac{p_\perp^2}{b} = \text{constant} \qquad\qquad \frac{p_\parallel b}{\rho} = \text{constant} \tag{A5}$$

where $p_\perp$ and $p_\parallel$ are the particle momenta (not pressure!) along and perpendicular to the magnetic field direction defined in the fluid frame. What does this imply for the relation between ΔP and the variation of b and ρ?

Start by changing momentum space coordinates to the adiabatic invariants

$$j_\perp \equiv \frac{p_\perp^2}{m} \qquad\qquad j_\parallel \equiv p_\parallel \left(\frac{m}{r} \right) \tag{A6}$$

where $r \equiv \rho/\rho_0$ and $m \equiv b/b_0$. The initial ($r = m = 1$) Maxwellian distribution function is

$$\frac{dn}{dj_\perp dj_\parallel} = \frac{n}{4\Theta K_2(1/\Theta)} e^{-\Gamma/\Theta} \tag{A7}$$

where

$$\Gamma = (1 + j_\perp + j_\parallel{}^2)^{1/2} \tag{A8}$$

and K_2 is the modified Bessel function of the second kind. This distribution is invariant under slow (adiabatic) changes in b and ρ.

Next, directly evaluate components of the pressure tensor using the kinetic theory definition (11) in the limit that r and m are close to 1:

$$\Delta P = P_\perp - P_\parallel = \int dj_\perp dj_\parallel \frac{dn}{dj_\perp dj_\parallel} \left(\frac{1}{2}\frac{p_\perp^2}{p^t} - \frac{p_\parallel^2}{p^t} \right), \tag{A9}$$

where the factor of $\frac{1}{2}$ arises because $p_\perp^2 = p_x^2 + p_y^2$ if b is aligned with z; here $p^t = \Gamma$ in the fluid frame. Notice that $p_\perp$ and $p_\parallel$ depend on r and m but the distribution function does not. Then

$$\left(\frac{d\Delta P}{d\tau} \right)_{ad} = \frac{d\Delta P}{dm}\frac{dm}{d\tau} + \frac{d\Delta P}{dr}\frac{dr}{d\tau} \tag{A10}$$

is the change in ΔP due to adiabatic deformation of the distribution function. If the anisotropy is small the derivative can be evaluated at $b = m = 1$.

The integrals needed for $d\Delta P/dm$ and $d\Delta P/dr$ can be evaluated analytically in the relativistic and non-relativistic limits.

In the non-relativistic limit $\Gamma \to 1 + \frac{1}{2}j_\perp + \frac{1}{2}j_\parallel^2$ and $dn/dj_\perp dj_\parallel \propto \Theta^{-3/2} \exp(-(j_\perp + j_\parallel^2))/(2\Theta)$. A few strengthening integrals later, one finds

$$\frac{1}{P}\left(\frac{d\Delta P}{d\tau}\right)_{ad} = 3\frac{dm}{d\tau} - 2\frac{dr}{d\tau} \tag{A11}$$

where $P = n\Theta$. Since the derivatives are evaluated at $m = r = 1$, $dm/d\tau = d\ln m/d\tau$, etc., and using the definition of m, r, find

$$\frac{1}{P}\left(\frac{d\Delta P}{d\tau}\right)_{ad} = \frac{d}{d\tau}\ln\left(\frac{b^3}{\rho^2}\right) \tag{A12}$$

as expected from the discussion in the preceding subsection.

In the ultrarelativistic limit $\Gamma \to (p_\perp^2 + p_\parallel^2)^{1/2} = (j_\perp + j_\parallel^2)^{1/2}$ at $b = m = 1$, and $dn/dj_\perp dj_\parallel \propto \Theta^{-3}\exp(-\Gamma/\Theta)$. The integrals give

$$\frac{1}{P}\left(\frac{d\Delta P}{d\tau}\right)_{ad} = \frac{12}{5}\frac{dm}{d\tau} - \frac{8}{5}\frac{dr}{d\tau}, \tag{A13}$$

or

$$\frac{1}{P}\left(\frac{d\Delta P}{d\tau}\right)_{ad} = \frac{4}{5}\frac{d}{d\tau}\ln\left(\frac{b^3}{\rho^2}\right). \tag{A14}$$

which differs by a factor of $4/5$ from the non-relativistic limit.

The rate of decay of the pressure anisotropy due to scattering is

$$\frac{1}{P}\left(\frac{d\Delta P}{d\tau}\right)_{scat} = -\omega\frac{\Delta P}{P}, \tag{A15}$$

which defines the scattering rate ω, so the total change

$$\frac{1}{P}\left(\frac{d\Delta P}{d\tau}\right) = \frac{1}{P}\left(\frac{d\Delta P}{d\tau}\right)_{ad} + \frac{1}{P}\left(\frac{d\Delta P}{d\tau}\right)_{scat}. \tag{A16}$$

In equilibrium scattering balances adiabatic forcing and, for the ultrarelativistic limit, the result is

$$\omega\frac{\Delta P}{P} = \frac{4}{5}\frac{d}{d\tau}\ln\left(\frac{b^3}{\rho^2}\right). \tag{A17}$$

We can do just a little bit more by evaluating the right hand side using the induction equation and the continuity equation, together with $b_\mu u^\mu = 0$. Along the way,

$$\frac{1}{2}\frac{d}{d\tau}\ln b^2 = -\nabla_\mu u^\mu + \hat{b}^\mu\hat{b}^\nu\nabla_\mu u_\nu \tag{A18}$$

and

$$\frac{d}{d\tau}\ln\rho = -\nabla_\mu u^\mu, \tag{A19}$$

so that

$$\frac{d}{d\tau}\ln\left(\frac{b^3}{\rho^2}\right) = 3\left(\hat{b}^\mu\hat{b}^\nu\nabla_\mu u_\nu - \frac{1}{3}\nabla_\mu u^\mu\right). \tag{A20}$$

Then in equilibrium

$$\Delta P = 3\rho\left[\frac{4}{5}\frac{\Theta}{\omega}\right]\left(\hat{b}^\mu\hat{b}^\nu\nabla_\mu u_\nu - \frac{1}{3}\nabla_\mu u^\mu\right). \tag{A21}$$

This is consistent with the non-relativistic derivation in equation A3, and, if we set

$$\nu = \frac{4}{5}\frac{\Theta}{\omega} \propto \tau c_s^2 \tag{A22}$$

with the relativistic thermodynamic derivation in §4 of the main text (in particular, equation 51 and surrounding results).

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